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Perpetuities & Annuities

An annuity is a stream of regularly-spaced cash flows. Examples of annuities include a standard fixed-rate mortgage, and the interest payments on a Treasury bond. The bond is the sum of an annuity and a lump-sum payment (i.e., the payment of principal on maturity).

We start by solving for the present value of a perpetuity. A perpetuity is a sequence of cash flows that never ends. The most common example is British government Console Bonds. Assume that the annual interest rate is r, and there are p compounding periods per year. Note that r must be on a p-period per year compounding basis. Let $z = \frac{r}{p}$. Then, z is the interest rate per compounding period.

By knowing how to obtain the present value of a future cash flow, we can value a perpetuity by adding all of the cash flows. That is, a perpetuity that pays \$1 every compounding period is worth PV when discounted at the rate r:

$$PV = \frac{1}{(1+z)} + \frac{1}{(1+z)^2} + \frac{1}{(1+z)^3} + \dots \tag{1}$$

Because this is an infinite sum, we can solve it using differences. Multiply both sides of this equation by $(1+z)$:

$$PV(1+z) = 1 + \frac{1}{(1+z)} + \frac{1}{(1+z)^2} + \frac{1}{(1+z)^3} + \dots \tag{2}$$

Now subtract (1) from (2):

$$PV(1+z) - PV = 1 \tag{3}$$

This leaves us the result that $PV = \frac{1}{z}$.

This is the formula for a perpetuity that pays \$1 every p periods per year into perpetuity, at the yield to maturity r. So for a perpetuity that makes a periodic payment of PMT the general formula for its present value (PV) is:

$$PV = \frac{PMT}{z} \tag{4}$$

Where PMT is paid p times per year into perpetuity, r is the perpetuity's yield to maturity on a p-period compounding basis, and $z = \frac{r}{p}$.
This means that if we know r and PMT we can solve for the perpetuity's present value. If we know the perpetuity's PMT and value we can solve for its yield. If we know r and the present value we can solve for the payment.

So now consider an annuity that expires in T years. This instrument pays \$1 p times per year for T years. Let $N = p \cdot T$ be the total number of payments in the annuity.

$$PV = \frac{1}{z} - \frac{\frac{1}{z}}{(1+z)^N} \tag{5}$$

That is the annuity is the same as a perpetuity that starts today minus that perpetuity that starts in T years. For an annuity that makes periodic payments of PMT, its value is:

$$PV = PMT \cdot \left\{ \frac{1}{z} - \frac{\frac{1}{z}}{(1+z)^N} \right\} \tag{6}$$

The term in brackets is called the annuity factor.
For example, consider a \$250,000 standard fixed rate mortgage with interest rate of 4%. What is the monthly payment? (A "standard" fixed rate mortgage in the US makes monthly payments over a 30-year term.)

$$250,000 = \frac{PMT}{.0033} - \frac{\frac{PMT}{.0033}}{(1+.0033)^{360}} \tag{7}$$

In this case the RHS of (4) is 209.46. So we have: $PMT = \frac{250,000}{209.46} = 1193.54$.