The CIR Model

Cox, Ingersoll, and Ross (1985) model an equilibrium, no-arbitrage economy, with a representative agent. From a theoretical point of view, the model links the data on yields to one or more latent independent factors. The model posits that the time series evolution of this latent factor is a mean-reverting, square-root process:

$$dz_j = \kappa_j(\theta_j - z_j)dt + \sigma_j\sqrt{z_j}d\omega_j,$$

where:

$j = 1, \cdots, J$ (the number of factors), and

ω_j is a Wiener process.

Bond prices depend on the current value of the state variable, as well as its expected evolution, along with a risk premium, λ. Specifically, the price of a τ-year bond, at time t is:

$$P_{t,t+\tau} = \prod_{j=1}^{J} \Lambda_{j,t,\tau}\; e^{-\beta_{j,t,\tau}\cdot z_{j,t}},$$

where:

$\Lambda_{j,t,\tau} = \left[\frac{2\gamma_j e^{[(\kappa_j+\lambda_j+\gamma_j)\tau]/2}}{(\kappa_j+\lambda_j+\gamma_j)(e^{\tau\gamma_j}-1)+2\gamma_j}\right]^{2\kappa_j\theta_j/\sigma_j^2}$

$\beta_{j,t,\tau} = \frac{2(e^{\tau\gamma_j}-1)}{(\kappa_j+\lambda_j+\gamma_j)(e^{\tau\gamma_j}-1)+2\gamma_j}$

$\gamma_j = ((\kappa_j+\lambda_j)^2 + 2\sigma_j^2)^{1/2}$.

For zero coupon bonds, the continuously compounded yield to maturity is:

$$R_{t,t+\tau} = \frac{\sum_{j=1}^{J}(\beta_{j,t,\tau}\cdot z_j - \log\Lambda_{j,t,\tau})}{\tau}.$$